Response to “Supplement.arXiv.2509.14820 (Laplace comparison etc)”

Xiuxiong Chen, Bing Wang

Oct 30, 2025

In the note [9], the authors appeal to three questions in [3]. **The first is the repeating of the pcr issue. The second is a typo, can be fixed replacing = by ≤. The third is an inaccuracy citation, can be fixed by deleting a redundant sentence.**

We shall respond them term by term in detail as follows.

(a). In their new note [9], the authors appeal to our definition of pcr (polarized canonical radius) and claims that it is inconsistent whether the **multiplicity** of line bundle is allowed to go to ∞. From line -4 to line -3 on page 2, and line -10 to line -7 of page 2 of [9], they write separately:

(A) *Therefore, this claim of the note on the* ***multiplicity*** *of the line bundles* ***going to infinity*** *already contradicts with the definition of the pcr, unless one changes the definition of pcr.*

(B) *However in Definition 3.10 on p36 of [3] of the pcr, which is and should be the same for all manifolds in the paper, the* ***multiplicity of the line bundle is bounded by a large but fixed number*** $2k_0$*. This constant* k_0 *comes from Proposition 3.9 two pages earlier and, as written, is independent of the multiplicity of the line bundle.*

The confusion stems from a misunderstanding of pcr. The multiplicity should be calculated with respect to correct line bundle.

Definition 0.1. *[3] For a polarized manifold* (M, g, J, L, h)*, we consider the quantity*

$$\bar{b}_{(g,L)}(x) := \sup_{1\leq j\leq 2k_0} b^{(j)}(x),$$

where $b^{(j)}$ *is the Bergman function of* L^j*. Here* $j \in \{1, 2, 3, \cdots, 2k_0\}$*, which is a bounded range. We say* $\mathbf{pcr}(x) \geq 1$ *if* $\mathbf{cr}(x) \geq 1$ *and* $\bar{b}_L(x) \geq -2k_0$.

The uniform constant k_0 leads to these authors the impression that **multiplicity** must be uniformly bounded. Naively speaking, one wish to define pcr by $\bar{b}_{(g,L)}(x) = b^{(1)}(x)$ directly and if so, then the authors in [8] would not have objection for the power of line bundle tends to ∞. For geometrical reasons, we need to use full spectrum of (M, g, J, L^j, h^j) with $j = 1, 2, \cdots 2k_0$ (See Proposition 6.2 of [7], which realized the necessity of considering more spectrum in the limit space, but made a mistake of the power, as pointed out by Example 5.8 of [6]. See also Theorem

1.1 of [5]). Note that if $j > 1$, then (M, g, J, L^j, h^j) is not polarized, as $-\sqrt{-1}\partial\bar{\partial}\log h^j$ is the metric form for $jg \neq g$.

One can certainly study pcr for rescaled version of polarized manifold with rescaling constant as large as one wished, even goes to ∞. For any integer $m > 1$, (M, mg, J, L^m, h^m) is a polarized manifold. Let $b^{'(j)}$ be the Bergman function of $(L^m)^j = L^{mj}$, with respect to underlying metric mg (**NOT** mjg) and line bundle metric h^{mj}, $\{S_l^{(mj)}\}_{l=0}^{N_{mj}}$ be orthonormal basis of $H^0(M, L^{mj})$. Then it follows from definition that

$$\bar{b}_{(mg,L^m)}(x) := \sup_{1\leq j\leq 2k_0} b^{'(j)}(x) = \sup_{1\leq j\leq 2k_0} \log\left\{\sum_{l=0}^{N_{mj}} \|S_l^{(mj)}\|^2_{h^{mj}}(x)\right\}.$$

So j is bounded but $mj \to \infty$. This explains why the scaling formula (4.3)-(4.5) on p.41 of [8], or (1.4)-(1.6) on p.2 of [9] is wrong. The missing scaling in the line bundle direction leads to their unfortunate mistakes.

(b) Tangent cones.

On page 3, line 18: the authors of [9] write:

*...**We are not sure why** 0 **appears** on the right hand of the identity 8 line below (3.57) on p46. ... For (3.56) itself, it is not explained why the left hand side should equal the negative of the infimum of the W-functional. Note the test function u_i, as written, is the heat kernel of the conjugate heat equation. The infimum of the W functional is not achieved by the heat kernel for general compact manifolds.*

The authors should read more carefully. **We have already explained explicitly the reason why 0 appears in the following sentences**, which we copy down here (exactly below the questioned 0):

It follows from Cheeger-Gromov convergence and the estimate (3.56) that

$$\int\int_{K\times[-H,0]} 2|t|\left|Ric + \nabla\nabla\hat{f} + \frac{\hat{g}}{2t}\right|^2 \hat{u}dvdt = 0.$$

Note the scaling invariance of $\hat{u}dv$ and $|t|\left|Ric + \nabla\nabla\hat{f} + \frac{\hat{g}}{2t}\right|^2 dt$. Actually, if the above equality fails for some K and H, then by definition of tangent space and the integral accumulation, we shall obtain the left had side of (3.56) is infinity and obtain a contradiction.

We have already highlight the keys why the equality holds. We believe they are **elementary** for experts in geometric analysis.

Here, the equation (3.56) is

$$\int_{-1}^{0}\int_{M_i} 2|t|\left|Ric_{g_i} + \nabla\nabla f_i + \frac{g_i}{2t}\right|^2 u_i dv_{g_i}dt = -\mu(M_i, g_i(t_i), 1) \leq C.$$

There is a **typo** here: the equality in the above line should be $\leq$, since

$$\int_{-1}^{0}\int_{M_i} 2|t|\left|Ric_{g_i} + \nabla\nabla f_i + \frac{g_i}{2t}\right|^2 u_i dv_{g_i}dt = -W(M_i, g_i(t_i), 1, f_i) \leq -\mu(M_i, g_i(t_i), 1) \leq C.$$

However, **this does not affect the following application at all.**

(c) Geodesics.

On page 3, line 29: the authors of [9] write:

Thirdly, it appears to us that in the proof of Proposition 3.39 on the connectivity of geodesics on p79 of [3], the words "curve" on line 2 should be replaced by "minimal geodesics". The reason is that this is the negation, in the method of proof by contradiction, of the pertinent condition 4) in the canonical radius in Definition 2.9 on p 18 of [3]. However, if one changes it to minimal geodesic, then one can not reach the stated contradiction in the last two lines of the proof since only a generic curve joining the two points y_i, z_i were constructed.

This confusion is caused by **inaccurate citation** when we separate our original manuscript [1], we thank the authors of [9] to point this out. **But this seems not an issue if the reader would like to take a minute to read the origin of our paper.** In section 2 of [3], we copy down many definitions in [2] (cf. page 14 of [3], at the beginning of section 2). Clearly, Definition 2.9 of [3] is nothing but Definition 3.5 of [2]. The last item of Definition 2.9 of [3] is

4) Connectivity estimate: $B(x_0, r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ is $\frac{1}{2}\epsilon_b r$-regularly connected on the scale r. Namely, every two points in $B(x_0, r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ can be connected by a shortest geodesic $\gamma \subset \mathcal{F}^{(r)}_{\frac{1}{2}\epsilon_b r}(M)$.

Comparing Definition 2.9 of [3] and Definition 3.5 of [2], the only difference is the following extra sentence

(*):Namely, every two points in $B(x_0, r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ can be connected by a shortest geodesic $\gamma \subset \mathcal{F}^{(r)}_{\frac{1}{2}\epsilon_b r}(M)$.

This sentence was added per request of the referee, as $\frac{1}{2}\epsilon_b r$-regularly connected is not explained if we do not copy down Definition 3.4 of [2], which we copy down here:

Definition 3.4. A subset Ω of M is called ϵ-regular-connected on the scale ρ if every two points $x, y \in \Omega$ can be connected by a rectifiable curve $\gamma \subset \mathcal{F}^{(\rho)}_{\epsilon}$ and $|\gamma| < 2d(x, y)$. For notational simplicity, if the scale is clear in the context, we shall just say Ω is ϵ-regular-connected.

Therefore, this inaccuracy can be fixed if we just **delete the sentence (*)** and quote Definition 3.4 of [2]. Also, it can be fixed by rewriting it as

*(**):Namely, every two points $\boldsymbol{x}, \boldsymbol{y}$ in $B(x_0, r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ can be connected by a **curve** $\gamma \subset \mathcal{F}^{(r)}_{\frac{1}{2}\epsilon_b r}(M)$ **satisfying** $|\gamma| < 2d(x, y)$.*

The proof of Proposition 3.39 on p79 is not affected at all.

In conclusion, all the three questions have nothing to do with serious mathematics.

References

[1] Xiuxiong Chen, Bing Wang, *Space of Ricci flows (II)*, arXiv: 1405.6797v4.

[2] Xiuxiong Chen, Bing Wang, *Space of Ricci flows (II)-Part A: Moduli of singular Calabi-Yau spaces*, Forum Math. Sigma 5 (2017), paper no. e32, 103pp.

[3] Xiuxiong Chen, Bing Wang, *Space of Ricci flows (II)-Part B: Weak compactness of the flows*, J. Differential Geom. 116 (2020), no. 1, 1-123.

[4] Xiuxiong Chen, Bing Wang, *Two Comments on Laplace comparison on Kähler Ricci flow and convergence (arXiv:2509.14820v1.)*

[5] S.K. Donaldson, S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math., 213(1): 63-106, 2014.

[6] Y. Odaka, C. Sport, S. Sun, *Compact moduli spaces of Del Pezzo surfaces and Kähler-Einstein metrics*, J. Differential Geom. 102 (2016), 127-172.

[7] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), 101-172.

[8] Gang Tian, Qi S. Zhang, Zhenlei Zhang, Meng Zhu, Xiaohua Zhu, *Laplace comparison on Kähler Ricci flow and convergence*, arXiv:2509.14820v1.

[9] Gang Tian, Qi S. Zhang, Zhenlei Zhang, Meng Zhu, Xiaohua Zhu, *Supplement.arXiv.2509.14820 (Laplace comparison etc)*, https://sites.google.com/view/qiszhang-ucr249/home, click Supplement.arxiv.2509.14820.