Two Comments on "Laplace comparison on Kähler Ricci flow and convergence (arXiv:2509.14820v1.)"

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(1). In the concluding acknowledgments of the recent preprint [3], one of the authors (Prof. Q. S. Zhang) thanks us for "helpful comments". This is confusing, as we had no opportunity to read or comment on the preprint prior to its posting. We note only that, more than six years ago, we exchanged emails with the same author addressing questions about our earlier paper.

(2). On pages 40-41 of [3], the authors challenge the definition of the *polarized canonical radius* (pcr) in [1], claiming–based on a direct computation–that the pcr cannot be bounded from below. Their claim, in essence, is the following (lines 8-21, page 41 of [3]):

Let (M, g, J, L, h) be a polarized Kähler manifold. Then we proceed by rescaling. Let $\tilde{g} = 4kj_0^2 g$ and $\{\tilde{S}_i^{(j)}\}$ be orthonormal basis of $H^0(M, L^j)$ under the metrics $\tilde{\omega}$ and $\tilde{h}$. Then

$$\inf_M \sum_{i=0}^{N_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}(t)}(x,t) \le \frac{1}{|M|_{\tilde{g}}} \int_M \sum_{i=0}^{N_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}(t)}(x,t) d\tilde{g} \le \frac{CN_j}{|M|_{\tilde{g}}} = \frac{CN_j}{(4kj_0^2)^{\frac{n}{2}}|M|_g} \to 0,$$

as $k \to \infty$. Thus, $\inf_M \tilde{b}^{(j)}(x,t) \to -\infty$.

The **error** in their argument occurs on line 14 of page 41 in reference [3], where the authors assume that the line bundle remains unchanged when working on the manifold $(M, \tilde{g})$. However, to satisfy the polarization condition, the line bundle must be appropriately adjusted to $\tilde{L}$. Specifically, if $\tilde{g} = 4kj_0^2 g$, then $\tilde{L} = L^{4kj_0^2}$. Accordingly, $\{\tilde{S}_i^{(j)}\}$ should be taken as an orthonormal basis of $H^0(M, \tilde{L}^j) = H^0(M, L^{4kj_0^2 j})$ with respect to $d\tilde{g} = (4kj_0^2)^m dg$ and $\tilde{h}^j = h^{4kj_0^2 j}$. Define

$$\tilde{N}_j := \dim H^0(M, L^{4kj_0^2 j}) - 1.$$

Running their argument with this correction yields

$$\inf_M \sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) \le \frac{1}{|M|_{\tilde{g}}} \int_M \sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) d\tilde{g} \le \frac{C\tilde{N}_j}{(4kj_0^2)^{\frac{n}{2}}|M|_g}.$$

However, as $k \to \infty$, by Riemann-Roch theorem ($m = \frac{n}{2} = \dim_{\mathbb{C}} M$), we have

$$\tilde{N}_j \sim \frac{c_1^m(L)}{m!}(4kj_0^2 j)^m + O(k^{m-1}) = |M|_g (4kj_0^2 j)^{\frac{n}{2}} + O(k^{\frac{n}{2}-1}).$$

It follows that

$$\lim_{k\to\infty} \frac{C\tilde{N}_j}{(4kj_0^2)^{\frac{n}{2}}|M|_g} = Cj^{\frac{n}{2}} > 0.$$

Therefore, one cannot conclude that

$$\inf_M \sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) \to 0, \quad \text{as } k \to \infty,$$

and consequently one cannot conclude that

$$\inf_M \tilde{b}^{(j)}(x,t) = \inf_M \log \sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) \to -\infty, \quad \text{as } k \to \infty,$$

as claimed in [3], line 22 on page 41.

Actually, one can show that the conclusion $\inf_M \tilde{b}^{(j)}(x,t) \to -\infty$ is impossible. Since $\{\tilde{S}_i^{(j)}\}$ is an orthonormal basis of $H^0(M, \tilde{L}^j) = H^0(M, L^{4kj_0^2 j})$ with respect to $d\tilde{g} = (4kj_0^2)^m dg$ and $\tilde{h}^j = h^{4kj_0^2 j}$, it is clear that $\{(4kj_0^2)^{\frac{m}{2}}\tilde{S}_i^{(j)}\}$ is an orthonormal basis of $H^0(M, \tilde{L}^j)$ with respect to dg and $\tilde{h}^j$. By the expansion formula of Bergman kernel (Yau-Tian-Zelditch-Lu-Catlin, cf. [5]), we have

$$\sum_{i=0}^{\tilde{N}_j} \|(4kj_0^2)^{\frac{m}{2}}\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) = \frac{(4kj_0^2 j)^m}{\pi^m} + O(k^{m-1})$$

which is the same as

$$\sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) = \frac{j^m}{\pi^m} + O(k^{-1}).$$

Therefore, for each fixed $x \in M$, we have

$$\tilde{b}^{(j)}(x,t) = \log \sum_{i=0}^{\tilde{N}_j} \|\tilde{S}_i^{(j)}\|^2_{\tilde{h}^j(t)}(x,t) = \log \frac{j^m}{\pi^m} + O(k^{-1}) \to \log \frac{j^m}{\pi^m} \neq -\infty, \quad \text{as } k \to \infty.$$

Since M is compact, the above asymptotic behavior excludes the possibility: $\inf_M \tilde{b}^{(j)}(x,t) \to -\infty$ as $k \to \infty$.

Updates: The authors of [3] recently update their paper and move the appendix to a new note [4], where they basically repeat their so called "counter-example". We put the further response in our new note [2].

References

[1] Xiuxiong Chen, Bing Wang, *Space of Ricci flows (II)-Part B: Weak compactness of the flows*, J. Differ. Geom. 116 (2020), no. 1, 1-123.

[2] Xiuxiong Chen, Bing Wang, *Response to "Supplement.arXiv.2509.14820 (Laplace comparison etc)"*, https://ustcwangbing.github.io/chinese/, in "further response" under the Preprints/Space of Ricci flows(II).

[3] Gang Tian, Qi S. Zhang, Zhenlei Zhang, Meng Zhu, Xiaohua Zhu, *Laplace comparison on Kähler Ricci flow and convergence*, arXiv:2509.14820v1.

[4] Gang Tian, Qi S. Zhang, Zhenlei Zhang, Meng Zhu, Xiaohua Zhu, *Supplement.arXiv.2509.14820 (Laplace comparison etc)*, https://sites.google.com/view/qiszhang-ucr249/home, click Supplement.arxiv.2509.14820.

[5] Steve Zelditch, *Szegö kernels and a theorem of Tian*, I.M.R.N., 1998, no. 6, 317-331.